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Deformation quantization of geometric quantum mechanics

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Abstract

Second quantization of a classical nonrelativistic one-particle system as a deformation quantization of the Schrödinger spinless field is considered. Under the assumption that the phase space of the Schrödinger field is $\mathbb{C}^\infty$, both the Weyl–Wigner–Moyal and Berezin deformation quantizations are discussed and compared. Then the geometric quantum mechanics is also quantized using the Berezin method under the assumption that the phase space is $\mathbb{C}P^\infty$ endowed with the Fubini–Study Kählerian metric. Finally, the Wigner function for an arbitrary particle state and its evolution equation are obtained. As is shown this new 'second quantization' leads to essentially different results than the former one. For instance, each state is an eigenstate of the total number particle operator and the corresponding eigenvalue is always $1/\hbar$.

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1. Introduction

Deformation quantization and, in general, noncommutative geometry have been the subjects of a great deal of renewed interest. In the deformation quantization approach the quantization is considered as a noncommutative deformation $\mathcal{A}_N$ of the algebra of the classical observables $\mathcal{A}_C$ in phase space [1, 2]. The resulting quantum algebra of linear operators is now equivalent to the deformation of the original algebra $\mathcal{A}_C$. This can be done at the level of the product of observables $\mathcal{O}_1 * \mathcal{O}_2$ or at the level of the Poisson–Lie bracket $\{\mathcal{O}_1, \mathcal{O}_2\}_*$. It depends on the type of the deformation quantization which is involved. In this paper we consider both types of situation. Here we deal with the Weyl–Wigner–Moyal deformation quantization [3–5] (for a recent review of this topic, see [6, 7]) and Berezin's deformation quantization [8–21].

For noncommutative geometry the situation is quite similar. In this case the deformed space is the noncommutative spacetime and the usual algebra of smooth functions is deformed into an associative and noncommutative algebra with the corresponding Moyal $*$-product. Yang–Mills gauge theories can be transformed into noncommutative gauge theories by replacing the usual matrix product by the Moyal $*$-product. These theories are strongly motivated since they can be obtained from the operator product expansion of string theories [22].

Weyl–Wigner–Moyal deformation quantization is very useful for the description of *flat* finite-dimensional phase spaces (or spacetimes) and many results have been obtained mainly by using this formalism. However, for the more general phase spaces (or spacetimes) further generalizations are required. One of them is the Fedosov deformation quantization [2] for an arbitrary symplectic manifold of finite dimension or Kontsevich's deformation quantization [23] for the case of a general finite-dimensional Poisson manifold. Another approach, extensively used in our paper, is the Berezin deformation quantization which is especially useful for Kählerian manifolds.

(It is worth noting the promising application of the Berezin formalism to the noncommutative sphere and noncommutative solitons on Kähler manifolds [24,25]. Recently, Berezin's deformation quantization has also been used to construct a nonperturbative formulation of quantum mechanics which includes S-duality symmetries observed in quantum theories of fields and strings. Such a formulation is based on a topological limit of the Berezin quantization of the upper half-plane [26]).

As there exist many definitions of what is meant by quantization (geometric quantization, Leray quantization, half-form quantization, Odzijewicz quantization, etc) we should explain now, with some precision, the approach to the problem of quantization used in this paper. We deal with deformation quantization in a formal sense given by Bayen *et al* [1] and Fedosov [2] and with Berezin's quantization [9, 10]. Let (M, ω) be a symplectic manifold and let $C^\infty(M)[[\hbar]]$ be a linear space of the formal power series

$$f(x;\hbar) = \sum_{k=0}^{\infty} \hbar^k f_k(x), \qquad x \in M \tag{1.1}$$

with $f_k \in C^\infty(M)$. Deformation quantization [1,2] is an associative algebra $(C^\infty(M)[[\hbar]], *)$, where $*$ is a product called the *star-product*, satisfying the following conditions:

(i) $f(x;\hbar) * g(x;\hbar) = \sum_{k=0}^{\infty} \hbar^k c_k(x)$, where the functions $c_k \in C^\infty(M)$ depend on f_i, g_j and their derivatives $\partial^r f_i, \partial^s g_j$ with $i + j + |r| + |s| \leqslant k$ (*locality*);
(ii) $c_0(x) = f_0(x) g_0(x)$;
(iii) $f * g - g * f = \mathrm{i}\hbar\{f_0, g_0\} + \mathrm{O}(\hbar^2)$ where $\{.,.\}$ represents the Poisson bracket.

In a sense the Berezin quantization is an example of the so-called strict deformation quantization where the formal series are assumed to be convergent. As is known and will be explained later on, the Berezin quantization is especially useful when the symplectic manifold has the Kähler structure.

The aim of this paper is to apply the Berezin approach to quantize geometric quantum mechanics and then to compare the result with the usual second quantization of the Schrödinger field. The geometric interpretation of quantum mechanics is a subject considered in the literature by a series of authors [27–35] and is based on the identification of the quantum phase space coming from the formal solution of the Schrödinger equation for a two-state system with the complex projective space $\mathbb{C}P^1 \cong S^2$. This does admit an immediate generalization to $\mathbb{C}P^n$ and in the general case we have to deal with $\mathbb{C}P^\infty$. All these spaces endowed with the well known Fubini–Study metrics are, of course, Kähler manifolds. (The geometric structure of $\mathbb{C}P^\infty$ as a Kähler manifold has been discussed, for example, by Kobayashi [36].) Moreover, the usual axiomatic formulation of quantum mechanics can be translated into a geometric language. For instance, the probability transition is given in terms of the Fubini–Study metric, while the quantum evolution equation is governed by the Kähler form.

Hence, it seems to be natural to consider the geometric quantum mechanics as a classical theory on the phase space (symplectic manifold $\mathbb{C}P^{\infty}$). The only essential difference between geometric quantum mechanics and other classical theories is that in the former not every real function on the phase space is an observable since each observable here must be the expected value of some Hermitian operator (see the formula (3.1)). Consequently, the product of two observables, in general, is no longer an observable. However, the Poisson bracket of these observables is still an observable (see section 3).

Now the question is if the quantization of this theory is equivalent to the usual second quantization. As is shown in this paper it is not so, and the quantization of geometric quantum mechanics leads to some new results which are not observed in the case of the second quantization.

Our paper is organized as follows. In section 2 we deal with the second quantization as a deformation quantization of the Schrödinger field. Assuming that the respective phase space is $\mathbb{C}^{\infty}$ we first use the Weyl–Wigner–Moyal formalism and then the Berezin one. Section 3 is devoted to a brief review of the geometric formulation of quantum mechanics following [27–35]. Here we provide the notation which will be used in the following sections. Sections 4 and 5 are the main parts of the paper. In section 4 the Berezin quantization of the geometric quantum mechanics is given and some physical results are obtained which are drastically different from the ones known in the usual second quantization. In section 5 we find the Wigner functions for the particle states. The von Neumann–Liouville evolution equation for an arbitrary Wigner function is also given. Final remarks (section 6) close the paper.

2. Second quantization as a deformation quantization

We deal with a nonrelativistic particle without spin in three dimensions which is moving under the potential $V(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$. The evolution equation is the usual Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \Psi(\boldsymbol{x}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V(\boldsymbol{x})\right)\Psi(\boldsymbol{x}, t). \tag{2.1}$$

As it is used in the second quantization procedure [37], the Schrödinger equation (2.1) is treated as the classical field equation which can be derived from the following action:

$$S = \int \mathrm{d}t\, L(t), \qquad L(t) = \int \mathrm{d}^3x\, \overline{\Psi}\left(\mathrm{i}\hbar\dot{\Psi} - V\Psi + \frac{\hbar^2}{2m}\Delta\Psi\right), \tag{2.2}$$

where $\dot{\Psi} = \frac{\partial\Psi}{\partial t}$. Then $\frac{\delta S}{\delta\overline{\Psi}} = 0$ is equivalent to the Schrödinger equation (2.1) and $\frac{\delta S}{\delta\Psi} = 0$ gives the complex conjugate of equation (2.1). The canonical momentum is defined by

$$\Pi(\boldsymbol{x}, t) = \frac{\delta L(t)}{\delta\dot{\Psi}(\boldsymbol{x}, t)} = \mathrm{i}\hbar\overline{\Psi}(\boldsymbol{x}, t). \tag{2.3}$$

For the fundamental Poisson brackets we obtain

$$\{\Psi(\boldsymbol{x}, t), \Pi(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x} - \boldsymbol{x}') \Longrightarrow \{\Psi(\boldsymbol{x}, t), \overline{\Psi}(\boldsymbol{x}', t)\} = \frac{1}{\mathrm{i}\hbar}\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{2.4}$$

Finally, the Hamiltonian is given by

$$H = \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}^3x\, \Pi\left(V\Psi - \frac{\hbar^2}{2m}\Delta\Psi\right). \tag{2.5}$$

The energy eigenfunctions of the particle are found from the equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi_k(\boldsymbol{x}) = \varepsilon_k\psi_k(\boldsymbol{x}), \tag{2.6}$$

where $\psi_k(\boldsymbol{x}, t) = \psi_k(\boldsymbol{x}, 0)\exp\{-\frac{\mathrm{i}}{\hbar}\varepsilon_k t\}$ with normalization $\int \mathrm{d}^3x\, \overline{\psi}_k\psi_{k'} = \delta_{kk'}$. We can now expand $\Psi(\boldsymbol{x}, t)$ in terms of those eigenfunctions

$$\begin{aligned} \Psi(\boldsymbol{x}, t) &= \sum_k \frac{1}{\sqrt{\hbar}} Z_k(t)\psi_k(\boldsymbol{x}) = \sum_k \frac{1}{\sqrt{\hbar}} Z_k \exp\{-\mathrm{i}\omega_k t\}\psi_k(\boldsymbol{x}), \\ Z_k &= \sqrt{\hbar}\int \overline{\psi}_k(\boldsymbol{x})\Psi(\boldsymbol{x})\,\mathrm{d}^3x, \end{aligned} \tag{2.7}$$

where $\omega_k = \varepsilon_k/\hbar$, $\overline{\psi}_k(\boldsymbol{x}) := \overline{\psi}_k(\boldsymbol{x}, 0)$ and $\Psi(\boldsymbol{x}) := \Psi(\boldsymbol{x}, 0)$. The Poisson brackets for the Z-variables can be found from (2.4) and (2.7) to be

$$\{Z_k, Z_{k'}\} = 0 = \{\overline{Z}_k, \overline{Z}_{k'}\}, \qquad \{Z_k, \overline{Z}_{k'}\} = -\mathrm{i}\delta_{kk'}. \tag{2.8}$$

Define oscillator variables

$$Q_k = \frac{1}{\sqrt{2}}(Z_k + \overline{Z}_k), \qquad P_k = \frac{\mathrm{i}}{\sqrt{2}}(\overline{Z}_k - Z_k) \tag{2.9}$$

which by (2.8) satisfy the algebra

$$\{Q_k, Q_{k'}\} = 0 = \{P_k, P_{k'}\}, \qquad \{Q_k, P_{k'}\} = \delta_{kk'}. \tag{2.10}$$

The Z-variables can be written in terms of the oscillator variables as follows:

$$Z_k = \frac{1}{\sqrt{2}}(Q_k + \mathrm{i}P_k), \qquad \overline{Z}_k = \frac{1}{\sqrt{2}}(Q_k - \mathrm{i}P_k). \tag{2.11}$$

Hence in terms of the oscillator variables the field function Ψ and its conjugate momentum Π are given by

$$\Psi(\boldsymbol{x}) = \frac{1}{\sqrt{2\hbar}}\sum_k (Q_k + \mathrm{i}P_k)\psi_k(\boldsymbol{x}), \qquad \Pi(\boldsymbol{x}) = \sqrt{\frac{\hbar}{2}}\sum_k (\mathrm{i}Q_k + P_k)\overline{\psi}_k(\boldsymbol{x}). \tag{2.12}$$

The time evolution of the system in terms of the oscillator variables is described by

$$\begin{aligned} Q_k(t) &= \frac{1}{\sqrt{2}}(Z_k(t) + \overline{Z}_k(t)) = Q_k\cos(\omega_k t) + P_k\sin(\omega_k t), \\ P_k(t) &= \frac{\mathrm{i}}{\sqrt{2}}(\overline{Z}_k(t) - Z_k(t)) = P_k\cos(\omega_k t) - Q_k\sin(\omega_k t). \end{aligned} \tag{2.13}$$

From (2.12) one quickly finds that

$$\begin{aligned} Q_k &= \sqrt{\frac{\hbar}{2}}\left(\int \mathrm{d}^3x\,\Psi(\boldsymbol{x})\overline{\psi}_k(\boldsymbol{x}) + \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}^3x\,\Pi(\boldsymbol{x})\psi_k(\boldsymbol{x})\right), \\ P_k &= \frac{1}{\sqrt{2\hbar}}\left(\int \mathrm{d}^3x\,\Pi(\boldsymbol{x})\psi_k(\boldsymbol{x}) - \mathrm{i}\hbar\int \mathrm{d}^3x\,\Psi(\boldsymbol{x})\overline{\psi}_k(\boldsymbol{x})\right). \end{aligned} \tag{2.14}$$

It seems to be natural to define the phase space of the system considered by $\mathcal{Z} = \{(Q_0, Q_1, \ldots, P_0, P_1, \ldots) : (Q_0, Q_1, \ldots, P_0, P_1, \ldots) \in \mathbb{R}^\infty \times \mathbb{R}^\infty\}$ endowed with the symplectic form

$$\omega = \sum_k \mathrm{d}P_k \wedge \mathrm{d}Q_k. \tag{2.15}$$

In terms of complex coordinates Z_k and $\overline{Z}_k$ the symplectic form ω reads

$$\omega = -\mathrm{i}\sum_k \mathrm{d}Z_k \wedge \mathrm{d}\overline{Z}_k \tag{2.16}$$

and one can easily recognize it as the Kähler form for $\mathbb{C}^{\infty}$. The Kähler potential $\mathcal{K}$ for this case is given by

$$\mathcal{K} = \sum_k Z_k \overline{Z}_k. \tag{2.17}$$

Writing

$$\begin{aligned} \mathrm{d}Q_k &= \sqrt{\frac{\hbar}{2}}\left(\int \mathrm{d}^3x\,\overline{\psi}_k(\boldsymbol{x})\delta\Psi(\boldsymbol{x}) + \frac{1}{\mathrm{i}\hbar}\int \mathrm{d}^3x\,\psi_k(\boldsymbol{x})\delta\Pi(\boldsymbol{x})\right),\\ \mathrm{d}P_k &= \frac{1}{\sqrt{2\hbar}}\left(\int \mathrm{d}^3x\,\psi_k(\boldsymbol{x})\delta\Pi(\boldsymbol{x}) - \mathrm{i}\hbar\int \mathrm{d}^3x\,\overline{\psi}_k(\boldsymbol{x})\delta\Psi(\boldsymbol{x})\right), \end{aligned} \tag{2.18}$$

we obtain

$$\omega = \int \mathrm{d}^3x\,\delta\Pi(\boldsymbol{x}) \wedge \delta\Psi(\boldsymbol{x}) = \mathrm{i}\hbar\int \mathrm{d}^3x\,\delta\Psi^*(\boldsymbol{x}) \wedge \delta\Psi(\boldsymbol{x}). \tag{2.19}$$

2.1. Weyl–Wigner–Moyal deformation quantization of the Schrödinger field

Now we are prepared to give the deformation quantization of the Schrödinger field. This can be done in a similar fashion to as in the case of classical fields [38–40]. First we deal with the Weyl–Wigner–Moyal deformation quantization.

Let $F_1 = F_1(Q, P)$ and $F_2 = F_2(Q, P)$ be two functions on the phase space $\mathcal{Z}$. The *Moyal $*$-product* is defined by

$$(F_1 * F_2)(Q, P) = F_1(Q, P)\exp\left\{\frac{\mathrm{i}\hbar}{2}\overleftrightarrow{\mathcal{P}}\right\}F_2(Q, P), \tag{2.20}$$

where $\overleftrightarrow{\mathcal{P}}$ is the Poisson operator

$$\begin{aligned} \overleftrightarrow{\mathcal{P}} &:= \sum_k\left(\frac{\overleftarrow{\partial}}{\partial Q_k}\frac{\overrightarrow{\partial}}{\partial P_k} - \frac{\overleftarrow{\partial}}{\partial P_k}\frac{\overrightarrow{\partial}}{\partial Q_k}\right) = \mathrm{i}\sum_k\left(\frac{\overleftarrow{\partial}}{\partial \overline{Z}_k}\frac{\overrightarrow{\partial}}{\partial Z_k} - \frac{\overleftarrow{\partial}}{\partial Z_k}\frac{\overrightarrow{\partial}}{\partial \overline{Z}_k}\right)\\ &= \int \mathrm{d}^3x\left(\frac{\overleftarrow{\delta}}{\delta\Psi(x)}\frac{\overrightarrow{\delta}}{\delta\Pi(x)} - \frac{\overleftarrow{\delta}}{\delta\Pi(x)}\frac{\overrightarrow{\delta}}{\delta\Psi(x)}\right). \end{aligned} \tag{2.21}$$

Employing (2.11) and (2.12) one can write the Hamiltonian (2.5) in the following form:

$$H = \tfrac{1}{2}\sum_k \omega_k(Q_k^2 + P_k^2) = \sum_k \omega_k \overline{Z}_k Z_k. \tag{2.22}$$

Then the Heisenberg equation reads

$$\dot{F} = \{F, H\}_M, \tag{2.23}$$

where $\{\cdot,\cdot\}_M$ stands for the *Moyal bracket*

$$\{F, G\}_M := \frac{1}{\mathrm{i}\hbar}(F * G - G * F). \tag{2.24}$$

It is an easy matter to define the Wigner function for any state and it can be done in analogous way as for other classical fields (compare with [39, 40]). For example, the Wigner function of the ground state is defined by

$$Z_k * \rho_0 = 0 \tag{2.25}$$

for all k. With the use of (2.20) and (2.21), equation (2.25) can also be written:

$$Z_k\rho_0 + \frac{\hbar}{2}\frac{\partial\rho_0}{\partial\overline{Z}_k} = 0. \tag{2.26}$$

This equation has the solution

$$\rho_0 \sim \exp\left(-\frac{2}{\hbar}\sum_k Z_k \overline{Z}_k\right) = \exp\left(-\frac{1}{\hbar}\sum_k (Q_k^2 + P_k^2)\right) = \exp\left(\frac{2\mathrm{i}}{\hbar}\int \mathrm{d}^3x\, \Psi(\boldsymbol{x})\Pi(\boldsymbol{x})\right). \tag{2.27}$$

Having given the Wigner function for the ground state one can easily construct Wigner functions for higher states. For example, the Wigner function for two particles, one of which is in the state k_1 and the second one in the state k_2, is given by

$$\rho_{k_1k_2} \sim \overline{Z}_{k_1}\overline{Z}_{k_2} * \rho_0 * Z_{k_2}Z_{k_1}. \tag{2.28}$$

It is well known that the Weyl–Wigner–Moyal deformation quantization arises from the Weyl correspondence. According to this correspondence if $\hat{F}$ is any operator acting in the Hilbert space of states then the Weyl symbol F_W of $\hat{F}$ is defined by

$$F_W(Q, P) = \mathrm{Tr}\{\hat{\Omega}(Q, P)\hat{F}\}, \tag{2.29}$$

where $\hat{\Omega}(Q, P)$ is the Stratonovich–Weyl quantizer which can be written in the following form:

$$\hat{\Omega}(Q, P) = \int \prod_m \mathrm{d}\xi_m \exp\left\{-\frac{\mathrm{i}}{\hbar}\sum_k \xi_k P_k\right\}\left|Q - \frac{\xi}{2}\right\rangle\left\langle Q + \frac{\xi}{2}\right| \tag{2.30}$$

or in terms of Ψ and Π as the following operator valued functional:

$$\hat{\Omega}[\Psi, \Pi] = \int \mathcal{D}\xi \exp\left\{-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x\, \xi(\boldsymbol{x})\Pi(\boldsymbol{x})\right\}\left|\Psi - \frac{\xi}{2}\right\rangle\left\langle \Psi + \frac{\xi}{2}\right|. \tag{2.31}$$

It is also known that if F_W and G_W are the Weyl symbols of the operators $\hat{F}$ and $\hat{G}$, respectively, then the Weyl symbol of the product $\hat{F}\hat{G}$ is given by $F_W * G_W$ (for details, see for example, [40]).

Now we are going to consider the Berezin deformation quantization of the Schrödinger field.

2.2. *Berezin deformation quantization of the Schrödinger field*

Consider the complex manifold $\mathbb{C}^{n+1}$ endowed with a Kähler metric

$$\mathrm{d}s^2 = \sum_{k,l=0}^{n} g_{k\bar{l}}(\mathrm{d}Z^k \otimes \mathrm{d}\overline{Z}^l + \mathrm{d}\overline{Z}^l \otimes \mathrm{d}Z^k), \tag{2.32}$$

which in terms of the Kähler potential $\mathcal{K} = \mathcal{K}(Z, \overline{Z})$ reads

$$g_{k\bar{l}} = \frac{\partial^2\mathcal{K}}{\partial Z^k \partial\overline{Z}^l}. \tag{2.33}$$

The corresponding symplectic form is given by

$$\omega = -\mathrm{i}\sum_{k,l=0}^{n} g_{k\bar{l}}\, \mathrm{d}Z^k \wedge \mathrm{d}\overline{Z}^l \tag{2.34}$$

and it induces a Poisson bracket on the functions of $C^\infty(\mathbb{C}^{n+1})$

$$\{f, g\} = \sum_{k,l=0}^{n} \omega^{\bar{l}k}\left(\frac{\partial f}{\partial\overline{Z}^l}\frac{\partial g}{\partial Z^k} - \frac{\partial f}{\partial Z^k}\frac{\partial g}{\partial\overline{Z}^l}\right), \tag{2.35}$$

where $\omega^{\bar{l}k}$ is the tensor inverse to the symplectic form, i.e. $\sum_{l=0}^{n}\omega^{j\bar{l}}\omega_{\bar{l}k}=\delta_k^j$. We describe now the Berezin quantization of the classical system on $\mathbb{C}^{n+1}$ endowed with a Kähler metric [9, 11, 21].

Let $\mathrm{d}\mu$ be the volume form on $\mathbb{C}^{n+1}$

$$\mathrm{d}\mu(Z,\overline{Z})=\left(\frac{\omega}{2\pi\hbar}\right)^n=\det(g_{i\bar{j}})\prod_{k=0}^{n}\frac{\mathrm{d}Z^k\wedge\mathrm{d}\overline{Z}^k}{2\pi\mathrm{i}\hbar}. \tag{2.36}$$

Denote now by $\mathcal{F}_\hbar$ the Hilbert space of the entire functions on $\mathbb{C}^{n+1}$, square summable with respect to the Gaussian measure $\exp\{-\frac{1}{\hbar}\mathcal{K}(Z,\overline{Z})\}\,\mathrm{d}\mu(Z,\overline{Z})$.

The inner product of two functions $f_1, f_2\in\mathcal{F}_\hbar$ is defined by

$$(f_1,f_2)=c(\hbar)\int_{\mathbb{C}^{n+1}}f_1(Z)\overline{f_2(Z)}\exp\left\{-\frac{1}{\hbar}\mathcal{K}(Z,\overline{Z})\right\}\mathrm{d}\mu(Z,\overline{Z}). \tag{2.37}$$

Let $\{f_k\}$, $k=1,\ldots$ define an arbitrary orthonormal basis in $\mathcal{F}_\hbar$ and let

$$\mathcal{B}(Z,\overline{V})=\sum_{k=1}f_k(Z)\overline{f_k(V)} \tag{2.38}$$

be the *Bergman kernel*. (From the physical point of view the Bergman kernel will correspond to the coherent states.) Then the holomorphic functions $\Phi_{\overline{V}}(Z):=\mathcal{B}(Z,\overline{V})$ parametrized by $\overline{V}\in\overline{\mathbb{C}}^{n+1}$, form a supercomplete system in $\mathcal{F}_\hbar$. (Note that the overbar means complex conjugation and not the closure of the set.) For any bounded operator $\hat{F}$ in $\mathcal{F}_\hbar$ one defines the following function:

$$F_B(Z,\overline{V})=\frac{(\hat{F}\Phi_{\overline{V}},\Phi_{\overline{Z}})}{(\Phi_{\overline{V}},\Phi_{\overline{Z}})}. \tag{2.39}$$

The function $F_B(Z,\overline{Z})\in C^\infty(\mathbb{C}^{n+1})$ is called the *covariant symbol* of the operator $\hat{F}$. Now if $F_B(Z,\overline{Z})$ and $G_B(Z,\overline{Z})$ are two covariant symbols of $\hat{F}$ and $\hat{G}$, respectively, then the covariant symbol of $\hat{F}\hat{G}$ is given by the *Berezin–Wick star product* $F_B *_B G_B$

$$\begin{aligned}(F_B *_B G_B)(Z,\overline{Z})&\\ &=c(\hbar)\int_{\mathbb{C}^{n+1}}F_B(Z,\overline{V})G_B(V,\overline{Z})\frac{\mathcal{B}(Z,\overline{V})\mathcal{B}(V,\overline{Z})}{\mathcal{B}(Z,\overline{Z})}\exp\left\{-\frac{1}{\hbar}\mathcal{K}(V,\overline{V})\right\}\mathrm{d}\mu(V,\overline{V})\\ &\times c(\hbar)\int_{\mathbb{C}^{n+1}}F_B(Z,\overline{V})G_B(V,\overline{Z})\exp\left\{\frac{1}{\hbar}\mathcal{K}(Z,\overline{Z};V,\overline{V})\right\}\mathrm{d}\mu(V,\overline{V})\end{aligned} \tag{2.40}$$

where $\mathcal{K}(Z,\overline{Z};V,\overline{V}):=\mathcal{K}(Z,\overline{V})+\mathcal{K}(V,\overline{Z})-\mathcal{K}(Z,\overline{Z})-\mathcal{K}(V,\overline{V})$ is called the *Calabi diastatic function*. One can also show that

$$\mathrm{Tr}\,\hat{F}=c(\hbar)\int_{\mathbb{C}^{n+1}}F_B(Z,\overline{Z})\mathcal{B}(Z,\overline{Z})\exp\left\{-\frac{1}{\hbar}\mathcal{K}(Z,\overline{Z})\right\}\mathrm{d}\mu(Z,\overline{Z}). \tag{2.41}$$

In order to specialize the Berezin deformation quantization to the case of the Schrödinger field we should assume that our complex space is infinite dimensional, i.e. we deal with $\mathbb{C}^\infty$ and the metric is given by $g_{k\bar{l}}=\delta_{kl}$. The Kähler function is therefore defined by (2.17). In this case $c(\hbar)=1$ and the orthonormal basis in the Hilbert space $\mathcal{F}_\hbar$ can be chosen to be the Fock basis

$$f_{(s_0,s_1,\ldots)}(Z)=\prod_l\frac{Z_l^{s_l}}{\sqrt{s_l!\hbar^{s_l}}}. \tag{2.42}$$

For the Bergman kernel (2.38) we now obtain

$$\mathcal{B}(Z,\overline{V})=\exp\left\{\frac{1}{\hbar}\sum_k Z_k\overline{V}_k\right\}=\exp\left\{\frac{1}{\hbar}\mathcal{K}(Z,\overline{V})\right\}. \tag{2.43}$$

Straightforward calculations show that the covariant symbol of the operator

$$\hat{F} = \sum_{j,k,l,m} F^{lm}_{jk} (\hat{a}^\dagger_l)^j (\hat{a}_m)^k, \tag{2.44}$$

where $\hat{a}^\dagger_l$ and $\hat{a}_m$ are the creation and annihilation operators, respectively, reads

$$F_B(Z, \overline{Z}) = \sum_{j,k,l,m} F^{lm}_{jk} (Z_l)^j (\overline{Z}_m)^k = F_{Wick}(\overline{Z}, Z). \tag{2.45}$$

Here $F_{Wick}(Z, \overline{Z})$ stands for the *Wick symbol* of the operator (2.44) [8, 9, 13] (note the order of the arguments $Z, \overline{Z}$).

As can be proved (see e.g. [9]) the relation between covariant F_B and the Weyl F_W symbols of operator $\hat{F}$ reads

$$\begin{aligned} F_W &= \mathcal{N} F_B(\overline{Z}, Z) = \mathcal{N} F_{Wick}(Z, \overline{Z}) \\ \mathcal{N} &= \exp\left\{-\frac{\hbar}{2}\sum_k \frac{\partial^2}{\partial Z_k \partial \overline{Z}_k}\right\} = \exp\left\{-\frac{\hbar}{4}\sum_k\left(\frac{\partial^2}{\partial Q_k^2} + \frac{\partial^2}{\partial P_k^2}\right)\right\} \\ &= \exp\left\{-\frac{\mathrm{i}\hbar}{2}\int \mathrm{d}^3x\, \frac{\delta^2}{\delta\Psi(\boldsymbol{x})\delta\Pi(\boldsymbol{x})}\right\}. \end{aligned} \tag{2.46}$$

Consequently, the Moyal and Berezin–Wick star products are related by

$$F * G = \mathcal{N}(\mathcal{N}^{-1} F *'_B \mathcal{N}^{-1} G) \qquad F *'_B G = \mathcal{N}^{-1}(\mathcal{N} F * \mathcal{N} G), \tag{2.47}$$

where

$$\begin{aligned} F(Z, \overline{Z}) *'_B G(Z, \overline{Z}) &:= c(\hbar)\int_{\mathbb{C}^{n+1}} F(V, \overline{Z}) G(Z, \overline{V}) \exp\left\{\frac{1}{\hbar}\mathcal{K}(Z, \overline{Z}; V, \overline{V})\right\} \mathrm{d}\mu(V, \overline{V}) \\ &= G(Z, \overline{Z}) *_B F(Z, \overline{Z}). \end{aligned} \tag{2.48}$$

In what follows the $*'_B$-product will also be called the *Berezin–Wick star product* $*'_B$.

3. Geometric quantum mechanics

As has been pointed out by many authors [27–35], quantum mechanics can be formulated as a geometric theory on a symplectic manifold. We would like to briefly explain this approach.

States of a quantum mechanic system are represented by rays in the associated infinite-dimensional Hilbert space $\mathcal{H}$. The expectation value of an observable $\hat{\hat{F}}$ in a state defined by the ket vector $|Z\rangle = |Z_0, Z_1, \ldots\rangle$ is given by

$$\langle\hat{\hat{F}}\rangle = \frac{\langle Z|\hat{\hat{F}}|Z\rangle}{\langle Z|Z\rangle}. \tag{3.1}$$

Henceforth we use double hat for operators in usual quantum mechanics and single hat for operators acting in the Hilbert space of field states. Expression (3.1) suggests that the space of rays in $\mathcal{H}$, i.e. the complex projective space $\mathbb{C}P^\infty$, represents the phase space of the system and the observables are the functions on $\mathbb{C}P^\infty$ of the form (3.1). The complex coordinates Z_k introduced in the previous section (see (2.7)) constitute the *homogeneous coordinates* of $\mathbb{C}P^\infty$. Let $\tilde{U}_j$ be a subset of $\mathbb{C}^\infty$ defined by $U_j = \{(Z_0, Z_1, \ldots) \in \mathbb{C}^\infty : Z_j \neq 0\}$. Then one can define the *inhomogeneous coordinates* on the respective coordinate neighbourhood $U_j \subset \mathbb{C}P^\infty$, where U_j is the projection of $\tilde{U}_j$ on $\mathbb{C}P^\infty$, as follows:

$$z^0_{(j)} = \frac{Z_0}{Z_j}, \qquad z^1_{(j)} = \frac{Z_1}{Z_j}, \qquad \ldots.$$

In terms of the coordinates Z or z, the observable $\langle\hat{\hat{F}}\rangle$ reads

$$\langle\hat{\hat{F}}\rangle = \frac{\sum_{k,l} F_{kl}\overline{Z}_k Z_l}{\sum_k Z_k\overline{Z}_k} = \frac{\sum_{k,l\neq j} F_{kl}\overline{z}^k_{(j)} z^l_{(j)} + \sum_{k\neq j}(F_{jk}z^k_{(j)} + F_{kj}\overline{z}^k_{(j)}) + F_{jj}}{1 + \sum_{k\neq j} z^k_{(j)}\overline{z}^k_{(j)}}, \tag{3.2}$$

where $F_{kl} := \langle\psi_k|\hat{\hat{F}}|\psi_l\rangle = \overline{F}_{lk}$ for all l, k.

In particular, from (2.22) and (3.2) we get for the Hamiltonian $\langle\hat{\hat{H}}\rangle$

$$\langle\hat{\hat{H}}\rangle = \frac{\sum_{k\neq j}\omega_k\overline{z}^k_{(j)}z^k_{(j)} + \omega_j}{(1 + |z_{(j)}|^2)}, \tag{3.3}$$

where $|z_{(j)}|^2 := \sum_{k\neq j}|z^k_{(j)}|^2$.

The quantum phase space $\mathbb{C}P^\infty$ can be endowed in a natural manner with a Riemannian metric. To this end consider two ket vectors $|Z\rangle$ and $|Z + \mathrm{d}Z\rangle$. The transition probability $p(|Z\rangle, |Z + \mathrm{d}Z\rangle)$ between $|Z\rangle$ and $|Z + \mathrm{d}Z\rangle$ is given by

$$p(|Z\rangle, |Z + \mathrm{d}Z\rangle) = \frac{\langle Z + \mathrm{d}Z|Z\rangle\langle Z|Z + \mathrm{d}Z\rangle}{\langle Z|Z\rangle\langle Z + \mathrm{d}Z|Z + \mathrm{d}Z\rangle}. \tag{3.4}$$

Simple calculations show that up to the second order in $\mathrm{d}Z$ the transition probability $p(|Z\rangle, |Z + \mathrm{d}Z\rangle)$ reads

$$\begin{aligned} p(|Z\rangle, |Z + \mathrm{d}Z\rangle) &= 1 - \frac{\langle \mathrm{d}Z|\mathrm{d}Z\rangle\langle Z|Z\rangle - \langle Z|\mathrm{d}Z\rangle\langle \mathrm{d}Z|Z\rangle}{|\langle Z|Z\rangle|^2} \\ &= 1 - \sum_{k,l}\frac{(\sum_m |Z_m|^2)\delta_{kl} - \overline{Z}_k Z_l}{\sum_m |Z_m|^2}\,\mathrm{d}Z_k\,\mathrm{d}\overline{Z}_l. \end{aligned} \tag{3.5}$$

The second term of the right-hand side of (3.5) can be written in terms of the inhomogeneous coordinates $z^k_{(j)}$ as follows:

$$\sum_{k,l}\frac{(\sum_m |Z_m|^2)\delta_{kl} - \overline{Z}_k Z_l}{\sum_m |Z_m|^2}\,\mathrm{d}Z_k\,\mathrm{d}\overline{Z}_l = \sum_{k,l\neq j}\frac{(1 + |z_{(j)}|^2)\delta_{kl} - \overline{z}^k_{(j)}z^l_{(j)}}{(1 + |z_{(j)}|^2)^2}\,\mathrm{d}z^k_{(j)}\,\mathrm{d}\overline{z}^l_{(j)}. \tag{3.6}$$

This suggests us to define the metric $\mathrm{d}s^2$ on the quantum phase space $\mathbb{C}P^\infty$ such that on any $U_j \in \mathbb{C}P^\infty$ $\mathrm{d}s^2$ is proportional to (3.6). For further correspondence between the usual second quantization and the deformation quantization of geometric quantum mechanics we take the metric $\mathrm{d}s^2$ to be of the form

$$\begin{aligned} \mathrm{d}s^2 &= \sum_{k,l\neq j} g_{k\bar{l}}(\mathrm{d}z^k_{(j)}\otimes\mathrm{d}\overline{z}^l_{(j)} + \mathrm{d}\overline{z}^l_{(j)}\otimes\mathrm{d}z^k_{(j)}), \\ g_{k\bar{l}} &= \frac{(1 + |z_{(j)}|^2)\delta_{kl} - \overline{z}^k_{(j)}z^l_{(j)}}{(1 + |z_{(j)}|^2)^2}, \qquad k, l \neq j. \end{aligned} \tag{3.7}$$

The above metric is up to a constant factor the well known Fubini–Study metric [41, 42] and $\mathbb{C}P^\infty$ endowed with this metric is a Kähler manifold. Then the $\mathrm{d}s^2$ can be defined on U_j in terms of the Kähler potential $\mathcal{K}$ as follows:

$$\begin{aligned} g_{k\bar{l}} &= \frac{\partial^2\mathcal{K}(z_{(j)}, \overline{z}_{(j)})}{\partial z^k_{(j)}\partial\overline{z}^l_{(j)}}, \\ \mathcal{K} &= \mathcal{K}(z_{(j)}, \overline{z}_{(j)}) = \ln(1 + |z_{(j)}|^2) = \ln\left(\sum_k z^k_{(j)}\overline{z}^k_{(j)}\right). \end{aligned} \tag{3.8}$$

It is easy to show that for any $p \in U_j \cap U_l$ of coordinates $z_{(j)}$ in U_j and $z_{(l)}$ in U_l the following transformation rule:

$$\mathcal{K}(z_{(j)}, \overline{z}_{(j)}) = \mathcal{K}(z_{(l)}, \overline{z}_{(l)}) + 2\ln|z^l_{(j)}| \tag{3.9}$$

holds.

The Kähler form Ω is defined by

$$\Omega = -\mathrm{i} \sum_{k,l\neq j} g_{k\bar{l}}\, \mathrm{d}z^k_{(j)} \wedge \mathrm{d}\overline{z}^l_{(j)} \tag{3.10}$$

for any j. Now we are going to define the symplectic form $\omega = \sum_{k,l\neq j} \omega_{k\bar{l}}\, \mathrm{d}z^k_{(j)} \wedge \mathrm{d}\overline{z}^l_{(j)}$ on the quantum phase space in such a way that for any function f the evolution equation reads

$$\dot{f} = \{f, \langle\hat{\hat{H}}\rangle\} = \sum_{k,l\neq j} \omega^{\bar{k}l} \left(\frac{\partial f}{\partial \overline{z}^k_{(j)}} \frac{\partial \langle\hat{\hat{H}}\rangle}{\partial z^l_{(j)}} - \frac{\partial \langle\hat{\hat{H}}\rangle}{\partial \overline{z}^k_{(j)}} \frac{\partial f}{\partial z^l_{(j)}} \right). \tag{3.11}$$

In particular, for $z^k_{(j)}$ and $\overline{z}^k_{(j)}$ we have

$$\dot{z}^k_{(j)} = -\sum_{l\neq j} \omega^{\bar{l}k} \frac{\partial \langle\hat{\hat{H}}\rangle}{\partial \overline{z}^l_{(j)}}, \qquad \dot{\overline{z}}^k_{(j)} = \sum_{l\neq j} \omega^{\bar{k}l} \frac{\partial \langle\hat{\hat{H}}\rangle}{\partial z^l_{(j)}}, \qquad k \neq j, \tag{3.12}$$

where $\langle\hat{\hat{H}}\rangle$ is given by (3.3). However, by direct calculations one obtains

$$\dot{z}^k_{(j)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Z_k}{Z_j}\right) = \mathrm{i}z^k_{(j)}(\omega_j - \omega_k), \qquad \dot{\overline{z}}^k_{(j)} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\frac{Z_k}{Z_j}}\right) = -\mathrm{i}\overline{z}^k_{(j)}(\omega_j - \omega_k). \tag{3.13}$$

(There is no summation over k!)

Comparing both expressions (3.12) and (3.13) we conclude that

$$\omega_{k\bar{l}} = -\mathrm{i}g_{k\bar{l}}. \tag{3.14}$$

Hence the symplectic form ω on the quantum phase space compatible with the evolution equation (3.11) is equal to the Kähler form Ω

$$\omega = \Omega. \tag{3.15}$$

This brief outline of the geometric quantum mechanics shows that from this point of view quantum mechanics can, in a sense, be treated as a classical theory on the infinite-dimensional phase space $\mathbb{C}P^\infty$. Therefore, it seems natural to look for quantization of this classical theory. One expects that such a quantization should be equivalent to the usual second quantization. But as we are going to demonstrate in the next section this is not so. This proof will be performed with the use of Berezin's deformation quantization on $\mathbb{C}P^n$ with $n \to \infty$.

Here an important comment is needed. The analogy between the geometric quantum mechanics and classical theory should be considered on the level of the Poisson–Lie algebra and not on the level of the usual product algebra of observables. This follows from the fact that the usual product of two observables $\langle\hat{\hat{F}}\rangle\langle\hat{\hat{G}}\rangle$ is in general no longer an observable in a sense that it cannot be represented in the form of (3.1). On the other hand, using the formula

$$\omega^{\bar{k}l} = \mathrm{i}g^{\bar{k}l} = \mathrm{i}(1 + |z|^2)(\delta^{kl} + \overline{z}^k z^l) \tag{3.16}$$

after straightforward calculations one can show that (compare with [31])

$$\{\langle\hat{\hat{F}}\rangle, \langle\hat{\hat{G}}\rangle\} = -\mathrm{i}\langle[\hat{\hat{F}}, \hat{\hat{G}}]\rangle, \tag{3.17}$$

which means that the Poisson bracket of two observables is also an observable. Hence, deformation quantization of the geometric quantum mechanics is a deformation of the Poisson–Lie algebra rather than a deformation of the usual product algebra. This is so at least in the case of linear quantum mechanics. The nonlinear case will be considered in a separate paper.

4. Berezin's quantization of geometric quantum mechanics

We deal with $\mathbb{C}P^n$ endowed with the metric (3.7) defined by the Kähler potential (3.8). Then the Kähler form Ω and the symplectic form ω are given by (3.10) and (3.15), respectively. First, in analogy to the case of the Berezin quantization on $\mathbb{C}^n$ considered in the section 2.2, we would like to define the corresponding Hilbert space $\mathcal{F}_\hbar$. But the obvious problem arises as the only entire function on $\mathbb{C}P^n$ is, according to the Liouville theorem, the constant function. So the natural idea is to consider $\mathcal{F}_\hbar$ as the space of sections $\mathrm{Sec}(\mathcal{L})$ of some complex line bundle $\mathcal{L}$ over $\mathbb{C}P^n$ which admits the local trivialization $U_j \times \mathbb{C}$ for any j, $j = 0, 1, \ldots, n$ [14, 15]. As the measure of the set $\mathbb{C}P^n - U_j$ is equal to zero for every j one can look for a scalar product in $\mathcal{F}_\hbar$ which by analogy to (2.37) should be defined as follows:

$$(f_1, f_2) = c(\hbar) \int_{U_j} f_{1(j)}(z_{(j)}) \overline{f_{2(j)}(z_{(j)})} \exp\left\{-\frac{1}{\hbar}\mathcal{K}(z_{(j)}, \overline{z_{(j)}})\right\} \mathrm{d}\mu(z_{(j)}, \overline{z_{(j)}}), \qquad \forall j, \tag{4.1}$$

where $f_1, f_2 \in \mathrm{Sec}(\mathcal{L})$, $f_{1(j)}$ and $f_{2(j)}$ are the local representations of f_1 and f_2, respectively, on U_j, and $\mathrm{d}\mu$ is the measure

$$\begin{aligned}\mathrm{d}\mu(z_{(j)}, \overline{z_{(j)}}) &= \left(\frac{\omega}{2\pi\hbar}\right)^n = \det(g_{i\bar{l}}) \prod_{k\neq j} \frac{\mathrm{d}z^k_{(j)} \wedge \mathrm{d}\overline{z}^k_{(j)}}{2\pi \mathrm{i}\hbar} \\ &= \exp\{-(n+1)\ln(1+|z_{(j)}|^2)\} \prod_{k\neq j} \frac{\mathrm{d}z^k_{(j)} \wedge \mathrm{d}\overline{z}^k_{(j)}}{2\pi \mathrm{i}\hbar}.\end{aligned} \tag{4.2}$$

Now using formula (3.9) we can quickly find that the definition of the scalar product (4.1) is independent of the index j if and only if the representations of the sections on U_j and U_l are related by

$$f_{(j)}(z_{(j)}) = (z^l_{(j)})^{\frac{1}{\hbar}} f_{(l)}(z_{(l)}) \tag{4.3}$$

on $U_j \cap U_l$. This rule of transformation makes sense only if $\frac{1}{\hbar} = N$ *where* N *is some positive integer* [10, 11, 14, 15]. We then assume that indeed it is so. Consequently our construction indicates that the line bundle $\mathcal{L}$ is defined by the transition functions

$$h_{jl} : U_j \cap U_l \to \mathbb{C}, \qquad h_{jl} = (z^l_{(j)})^{\frac{1}{\hbar}}. \tag{4.4}$$

Therefore

$$\mathcal{L} = \otimes^{\frac{1}{\hbar}} (U_{1,n+1})^{-1}, \tag{4.5}$$

where $U_{1,n+1}$ is the *universal complex line bundle over* $\mathbb{C}P^n$ [42]. Then the Hilbert space is defined by

$$\mathcal{F}_\hbar = \mathrm{Sec}(\mathcal{L}). \tag{4.6}$$

(For detailed analysis of this construction see [14, 15].)

As the forthcoming calculations will be performed in the open set U_0 for simplicity we use the natural abbreviations by omitting the lower index (0). So for example, we write $z^k := z^k_{(0)}$, $f := f_{(0)}$, etc. First, let us compute the factor $c(\hbar)$ which appears in the definition of the scalar product (4.1). To this end one assumes that the norm of the cross section of the bundle $\mathcal{L}$ which on U_0 is represented by the unity function $f(z) = 1$ is equal to 1. So substituting (3.8) and (4.2) into (4.1) and also taking that $f_1(z) = f_2(z) = 1$ we obtain

$$1 = c(\hbar) \int_{U_0} \frac{1}{(1+|z|^2)^{\frac{1}{\hbar}+n+1}} \prod_{k=1}^{n} \frac{\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^k}{2\pi \mathrm{i}\hbar}. \tag{4.7}$$

The integral in (4.7) can be evaluated (see [43], the integral 4.638-3) to give

$$\int_{U_0} \frac{1}{(1+|z|^2)^{\frac{1}{\hbar}+n+1}} \prod_{k=1}^{n} \frac{\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^k}{2\pi \mathrm{i}\hbar} = \hbar^{-n} \frac{\Gamma(\frac{1}{\hbar}+1)}{\Gamma(\frac{1}{\hbar}+n+1)}.$$

Introducing this result into (4.7) one finds

$$c(\hbar) = \hbar^{n} \frac{\Gamma(\frac{1}{\hbar}+n+1)}{\Gamma(\frac{1}{\hbar}+1)}. \tag{4.8}$$

Remember that $\frac{1}{\hbar} = N \in \mathbb{Z}_{+}$. One can check that for any monomial $f(z)$ on U_0 of degree greater than $\frac{1}{\hbar}$ the integral

$$c(\hbar) \int_{U_0} |f(z)|^2 \frac{1}{(1+|z|^2)^{\frac{1}{\hbar}+n+1}} \prod_{k=1}^{n} \frac{\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^k}{2\pi \mathrm{i}\hbar}$$

diverges. It means that for $n < \infty$ the dimension of the Hilbert space $\mathcal{F}_{\hbar}$ is finite. To proceed further, especially to find the Bergman kernel, we need an orthonormal basis of $\mathcal{F}_{\hbar}$. One expects that an orthonormal basis of $\mathcal{F}_{\hbar}$ can be constituted by the sections of the line bundle $\mathcal{L}$ such that on U_0 they are represented by monomials of degree no greater than $\frac{1}{\hbar}$. Therefore, consider the monomials on U_0 of the following form:

$$e_{(s_1,\ldots,s_n)}(z) = \alpha_{(s_1,\ldots,s_n)} (z^1)^{s_1} \ldots (z^n)^{s_n}, \qquad s_1 + \cdots + s_n \leqslant \frac{1}{\hbar},\ s_1, \ldots, s_n \geqslant 0,$$

where $\alpha_{(s_1,\ldots,s_n)}$ is some positive factor. By straightforward calculations, also employing the formulae (obtained by *Mathematica*)

$$\sum_{k=0}^{s} \binom{s}{k} \Gamma(k+\tfrac{1}{2}) \Gamma(s-k+\tfrac{1}{2}) = \pi s!$$

and

$$\sum_{k=0}^{s} \sum_{l=0}^{r} (-1)^m \binom{s}{k} \binom{2r}{2l} \Gamma(s+l-k+\tfrac{1}{2}) \Gamma(r+k-l+\tfrac{1}{2}) = 0$$

one gets

$$(e_{(s_1,\ldots,s_n)}, e_{(s'_1,\ldots,s'_n)}) = \delta_{s_1 s'_1} \ldots \delta_{s_n s'_n} \Longleftrightarrow \alpha_{(s_1,\ldots,s_n)} = \sqrt{\frac{\frac{1}{\hbar}!}{s_1! \ldots s_n! (\frac{1}{\hbar} - \sum_{k\neq 0} s_k)!}}. \tag{4.9}$$

Hence the set

$$\left\{ e_{(s_1,\ldots,s_n)}(z) = \sqrt{\frac{\frac{1}{\hbar}!}{s_1! \ldots s_n! (\frac{1}{\hbar} - \sum_{k\neq 0} s_k)!}} (z^1)^{s_1} \ldots (z^n)^{s_n} \right\}_{s_1+\cdots+s_n \leqslant \frac{1}{\hbar}}$$

represents in U_0 an orthonormal basis of $\mathcal{F}_{\hbar} : \{e_{(s_1,\ldots,s_n)}\}_{s_1+\cdots+s_n \leqslant \frac{1}{\hbar}}$.

Now we are in a position to define the Bergman kernel $\mathcal{B}$ which in fact should be a global section of the bundle $\mathcal{L} \otimes \overline{\mathcal{L}}$ over $\mathbb{C}P^n \times \overline{\mathbb{C}P^n}$. By analogy with (2.38) the representation of the Bergman kernel $\mathcal{B} \in \mathrm{Sec}(\mathcal{L} \otimes \overline{\mathcal{L}})$ on $U_0 \times \overline{U_0}$ is defined by

$$\begin{aligned} \mathcal{B}(z, \overline{v}) &= \sum_{s_1+\cdots s_n \leqslant \frac{1}{\hbar}} e_{(s_1,\ldots,s_n)}(z) \overline{e_{(s_1,\ldots,s_n)}(v)} \\ &= \frac{\frac{1}{\hbar}!}{s_1! \ldots s_n! (\frac{1}{\hbar} - \sum_{k\neq 0} s_k)!} (z^1)^{s_1} \ldots (z^n)^{s_n} (\overline{v}^1)^{s_1} \ldots (\overline{v}^n)^{s_n} = \left(1 + \sum_{k=1}^{n} z^k \overline{v}^k\right)^{\frac{1}{\hbar}}. \end{aligned} \tag{4.10}$$

One quickly finds that the representation of $\mathcal{B}$ on any $U_j \times \overline{U_l}$, $j, l = 0, \ldots, n$, reads

$$\mathcal{B}(z_{(j)}, \overline{v}_{(l)}) = \left(\sum_{k=0}^{n} z^k_{(j)} \overline{v}^k_{(l)} \right)^{\frac{1}{\hbar}}. \tag{4.11}$$

Consequently the holomorphic functions $\Phi_{\overline{v}}(z) = \mathcal{B}(z, \overline{v})$ on U_0 parametrized by $\overline{v} \in \overline{U_0}$ represent a supercomplete system in the Hilbert space $\mathcal{F}_\hbar$, i.e. the set $\{\Phi_{\overline{v}} \in \mathcal{F}_\hbar\}_{\overline{v}\in\overline{U_0}}$ such that $(f, \Phi_{\overline{v}}) = f(v)$, for all $f \in \mathcal{F}_\hbar$. Note that in analogous way one can find a supercomplete system in $\mathcal{F}_\hbar$ parametrized by the points of any $\overline{U_l}$. This supercomplete system $\{\Phi_{\overline{v}_{(l)}} \in \mathcal{F}_\hbar\}_{\overline{v}_{(l)}\in\overline{U_l}}$ is defined in terms of its representation $\{\Phi_{\overline{v}_{(l)}}(z_{(j)})\}_{\overline{v}_l\in\overline{U_l}}$ on U_j by

$$\Phi_{\overline{v}_{(l)}}(z_{(j)}) := \mathcal{B}(z_{(j)}, \overline{v}_{(l)}) = \left(\sum_{k=0}^{n} z^k_{(j)}\overline{v}^k_{(l)}\right)^{\frac{1}{\hbar}}. \tag{4.12}$$

Now we have $(f, \Phi_{\overline{v}_{(l)}}) = f_{(l)}(v_{(l)})$, for all $f \in \mathcal{F}_\hbar$. The following relation holds:

$$\begin{aligned} \Phi_{\overline{v}_{(l)}}(z_{(j)}) &= \mathcal{B}(z_{(j)}, \overline{v}_{(l)}) = \exp\left\{\frac{1}{\hbar}\mathcal{K}(z_{(j)}, \overline{v}_{(l)})\right\}, \\ \mathcal{K}(z_{(j)}, \overline{v}_{(l)}) &= \ln\left\{\sum_{k=0}^{n} z^k_{(j)}\overline{v}^k_{(l)}\right\}. \end{aligned} \tag{4.13}$$

This relation is known as Berezin's hypothesis A [9].

Now we intend to define the covariant symbols of operators acting on $\mathcal{F}_\hbar$.

Let $\hat{F} : \mathcal{F}_\hbar \to \mathcal{F}_\hbar$ be a linear operator on $\mathcal{F}_\hbar$. (As for $n < \infty$ the dimension of $\mathcal{F}_\hbar$ is finite, every linear operator is also bounded.) Consider the functions $F_B(z_{(j)}, \overline{v}_{(l)})$

$$F_B(z_{(j)}, \overline{v}_{(l)}) := \frac{(\hat{F}\Phi_{\overline{v}_{(l)}}, \Phi_{\overline{z}_{(j)}})}{(\Phi_{\overline{v}_{(l)}}, \Phi_{z_{(j)}})}, \tag{4.14}$$

for all j, l. These functions are holomorphic on dense subsets $S_{j\bar{l}} \subset U_j \times \overline{U_l}$ which consist of all points $U_j \times \overline{U_l}$ such that $(\Phi_{\overline{v}_{(l)}}, \Phi_{z_{(j)}}) \neq 0$. Moreover, for any $(p, \overline{q}) \in S_{j\bar{l}} \cap S_{k\overline{m}}$ we have

$$F_B(z_{(j)}, \overline{v}_{(l)}) = F_B(z_{(k)}, \overline{v}_{(m)}),$$

where $(z_{(j)}, \overline{v}_{(l)})$ and $(z_{(k)}, \overline{v}_{(m)})$ are the respective coordinates of $(p, \overline{q})$. It means that the set of functions given by (4.14) defines a holomorphic function $F_B : \cup_{j,l} S_{j\bar{l}} \to \mathbb{C}$. Observe that $\cup_{j,l} S_{j\bar{l}}$ is a dense subset of $\mathbb{C}P^n \times \overline{\mathbb{C}P^n}$. The restriction of the function F_B to the points $\overline{q} = \overline{p}$ gives an analytic function with respect to the real structure on $\mathbb{C}P^n$ and is called the *covariant symbol* of the operator $\hat{F}$. Locally we have

$$F_B(z_{(j)}, \overline{z}_{(j)}) := \frac{(\hat{F}\Phi_{\overline{z}_{(j)}}, \Phi_{\overline{z}_{(j)}})}{(\Phi_{\overline{z}_{(j)}}, \Phi_{z_{(j)}})}, \tag{4.15}$$

for all j.

Letting $f \in \mathcal{F}_\hbar$ be represented in U_0 by $f(z)$ and letting $\hat{F}$ be a linear operator in $\mathcal{F}_\hbar$ we then have

$$\begin{aligned} (\hat{F}f)(z) &= (\hat{F}f, \Phi_{\overline{z}}) = (f, \hat{F}^\dagger\Phi_{\overline{z}}) \\ &= c(\hbar)\int_{U_0} (f, \Phi_{\overline{v}})(\Phi_{\overline{v}}, \hat{F}^\dagger\Phi_{\overline{z}}) \exp\left\{-\frac{1}{\hbar}\mathcal{K}(v, \overline{v})\right\} \mathrm{d}\mu\,(v, \overline{v}) \\ &= c(\hbar)\int_{U_0} (\hat{F}\Phi_{\overline{v}}, \Phi_{\overline{z}}) f(v) \exp\left\{-\frac{1}{\hbar}\mathcal{K}(v, \overline{v})\right\} \mathrm{d}\mu\,(v, \overline{v}) \\ &= c(\hbar)\int_{U_0} F_B(z, \overline{v}) f(v)\Phi_{\overline{v}}(z) \exp\left\{-\frac{1}{\hbar}\mathcal{K}(v, \overline{v})\right\} \mathrm{d}\mu\,(v, \overline{v}) \\ &= c(\hbar)\int_{U_0} \frac{F_B(z, \overline{v}) f(v)(1 + z\overline{v})^{\frac{1}{\hbar}}}{(1 + v\overline{v})^{\frac{1}{\hbar}+n+1}} \prod_{k=1}^{n} \frac{\mathrm{d}v^k \wedge \mathrm{d}\overline{v}^k}{2\pi\mathrm{i}\hbar}. \end{aligned} \tag{4.16}$$

Straightforward calculations lead to the following formula for the trace of an operator $\hat{F}$:

$$\mathrm{Tr}\,\hat{F} = c(\hbar)\int_{U_0} F_B(z,\overline{z})\,\mathrm{d}\mu\,(z,\overline{z}) =: \mathrm{Tr}\,F_B. \tag{4.17}$$

From the definition of the covariant symbol it follows immediately that for the unit operator $\hat{F} = \hat{1}$, its covariant symbol is the unit function. Using this fact in equation (4.17) one finds that the dimension of the Hilbert space $\mathcal{F}_\hbar$ reads

$$\dim \mathcal{F}_\hbar = \mathrm{Tr}\,\hat{1} = c(\hbar)\int_{U_0} \mathrm{d}\mu\,(z,\overline{z}) = \frac{\Gamma(\frac{1}{\hbar}+n+1)}{\Gamma(\frac{1}{\hbar}+1)\Gamma(n+1)} = \binom{\frac{1}{\hbar}+n}{\frac{1}{\hbar}}. \tag{4.18}$$

Finally, if $F_B(z,\overline{z})$ and $G_B(z,\overline{z})$ represent on U_0 the covariant symbols of the operators $\hat{F}$ and $\hat{G}$ respectively, then the covariant symbol of $\hat{F}\hat{G}$ is given by the *Berezin–Wick star product* $F_B *_B G_B$ which on U_0 is represented by

$$\begin{aligned}(F_B *_B G_B)(z,\overline{z}) &= c(\hbar)\int_{U_0} F_B(z,\overline{v})G_B(v,\overline{z})\frac{\mathcal{B}(z,\overline{v})\mathcal{B}(v,\overline{z})}{\mathcal{B}(z,\overline{z})}\exp\left\{-\frac{1}{\hbar}\mathcal{K}(v,\overline{v})\right\}\mathrm{d}\mu\,(v,\overline{v})\\ &= c(\hbar)\int_{U_0} F_B(z,\overline{v})G_B(v,\overline{z})\exp\left\{\frac{1}{\hbar}\mathcal{K}(z,\overline{z};v,\overline{v})\right\}\mathrm{d}\mu\,(v,\overline{v}),\end{aligned} \tag{4.19}$$

where $\mathcal{K}(z,\overline{z};v,\overline{v}) := \mathcal{K}(z,\overline{v}) + \mathcal{K}(v,\overline{z}) - \mathcal{K}(z,\overline{z}) - \mathcal{K}(v,\overline{v})$ is the Calabi diastatic function. As $\mathcal{K}(z,\overline{z};v,\overline{v}) = \mathcal{K}(v,\overline{v};z,\overline{z})$ from (4.17) and (4.19) it follows that

$$\mathrm{Tr}(F_B *_B G_B) = \mathrm{Tr}(G_B *_B F_B). \tag{4.20}$$

Hence, the Berezin–Wick star product is a *closed star product* [2, 44].

In order to perform a quantization of the geometric quantum mechanics we must work with $\mathbb{C}P^\infty$ [12, 36]. This can be done by taking the limit $n \to \infty$, but one should be careful because some objects might have no sense at all. For example

$$\lim_{n\to\infty} c(\hbar) = \infty.$$

The first important result in the case of $\mathbb{C}P^\infty$ is that we still have $\frac{1}{\hbar} = N \in \mathbb{Z}_+$. Then from (4.18) with $n \to \infty$ one gets

$$\dim \mathcal{F}_\hbar = \infty. \tag{4.21}$$

The orthonormal basis of $\mathcal{F}_\hbar$ for $n \to \infty$ can be chosen analogously as before and it is represented in U_0 by the set of monomials

$$\left\{e_{(s_1,s_2,\ldots)}(z) = \sqrt{\frac{\frac{1}{\hbar}!}{(\frac{1}{\hbar} - \sum_{k\neq 0} s_k)!}}\prod_{k\neq 0}\frac{(z^k)^{s_k}}{\sqrt{s_k!}}\right\}_{\sum_{k\neq 0} s_k \leqslant \frac{1}{\hbar}}.$$

As we know from the previous section the general observable on $\mathbb{C}P^\infty$ has the form given by (3.2). It seems to be natural to identify this observable with the Wick symbol of the respective operator $\hat{F}$ acting on $\mathcal{F}_\hbar$. So employing (4.16) and the relation

$$\langle\hat{\hat{F}}\rangle(z,\overline{z}) = F_{Wick}(z,\overline{z}) = F_B(\overline{z},z) \tag{4.22}$$

we have in U_0

$$
\begin{aligned}
(\hat{F}f)(z) &= c(\hbar)\int_{U_0} \frac{\sum_{k,l=1}^{n} F_{kl} z^k \overline{v}^l + \sum_{k=1}^{n}(F_{0k}\overline{v}^k + F_{k0} z^k) + F_{00}}{1 + z\overline{v}} \\
&\quad \times f(v)(1 + z\overline{v})^{\frac{1}{\hbar}} (1 + v\overline{v})^{-\frac{1}{\hbar}} \, \mathrm{d}\mu\,(v, \overline{v}) \\
&= \lim_{n\to\infty}\Bigg[\Bigg(\sum_{k=1}^{n} F_{k0} z^k + F_{00}\Bigg)\hbar \lim_{x\to 1}\frac{\partial}{\partial x}\Bigg(x^{\frac{1}{\hbar}} c(\hbar)\int_{U_0} f(v)\Bigg(1 + \frac{z\overline{v}}{x}\Bigg)^{\frac{1}{\hbar}} \\
&\quad \times (1 + v\overline{v})^{-\frac{1}{\hbar}} \, \mathrm{d}\mu\,(v, \overline{v})\Bigg) + \hbar \sum_{l=1}^{n}\Bigg(\sum_{k=1}^{n} F_{kl} z^k + F_{0l}\Bigg)\frac{\partial f}{\partial z^l}\Bigg] \\
&= \Bigg\{\Bigg(F_{00} + \sum_{k=1}^{\infty} F_{k0} z^k\Bigg) + \hbar \sum_{l=1}^{\infty}\Bigg(F_{0l} - F_{00} z^l + \sum_{k=1}^{\infty}(F_{kl} - F_{k0} z^l) z^k\Bigg)\frac{\partial}{\partial z^l}\Bigg\} f(z),
\end{aligned}
\tag{4.23}
$$

where $z\overline{v} := \sum_{k=1}^{\infty} z^k \overline{v}^k$ and we have also used the formula $(f, \Phi_{\overline{z}}) = f(z)$.

In particular, substituting (3.3) into (4.23) one gets the Hamilton operator in the following form:

$$
(\hat{H}f)(z) = \Bigg[\hbar \sum_{k=1}^{\infty}(\omega_k - \omega_0) z^k \frac{\partial}{\partial z^k} + \omega_0\Bigg] f(z) = \Bigg(\hbar \sum_{k=0}^{\infty} \omega_k \hat{a}_k^{\dagger} \hat{a}_k\Bigg) f(z) \tag{4.24}
$$

where

$$
\hat{a}_k^{\dagger}\hat{a}_k = z^k \frac{\partial}{\partial z^k}, \qquad \hat{a}_0^{\dagger}\hat{a}_0 = \frac{1}{\hbar} - \sum_{k=1}^{\infty} z^k \frac{\partial}{\partial z^k}. \tag{4.25}
$$

Simple calculations show that the operators defined by (4.25) can be extended to the whole Hilbert space $\mathcal{F}_\hbar$ giving the particle number operators $\hat{N}_k := \hat{a}_k^{\dagger}\hat{a}_k$ for $k = 0, 1, \ldots$. *However, it is not possible to define in $\mathcal{F}_\hbar$ the annihilation $\hat{a}_k$ and creation $\hat{a}_k^{\dagger}$ operators*. This is so because one cannot globally extend the operators of the form $\frac{\partial}{\partial z^k}$ and z^k. Using (4.15) and (4.22) one quickly finds that on U_0 (in what follows we omit the subindex Wick to denote the Wick symbol of an operator!)

$$
\begin{aligned}
N_k(z, \overline{z}) &= \frac{1}{\hbar}\frac{z^k \overline{z}^k}{1 + z\overline{z}}, \qquad k \neq 0, \\
N_0(z, \overline{z}) &= \frac{1}{\hbar}\frac{1}{1 + z\overline{z}}.
\end{aligned}
\tag{4.26}
$$

The vectors $e_{(s_1, s_2, \ldots)} \in \mathcal{F}_\hbar$ are eigenvectors of the operators $\hat{N}_k$

$$
\begin{aligned}
\hat{N}_k e_{(s_1, s_2, \ldots)} &= s_k e_{(s_1, s_2, \ldots)}, \qquad k \neq 0, \\
\hat{N}_0 e_{(s_1, s_2, \ldots)} &= \Bigg(\frac{1}{\hbar} - \sum_{k\neq 0} s_k\Bigg) e_{(s_1, s_2, \ldots)}.
\end{aligned}
\tag{4.27}
$$

From (4.25) it follows that the total particle number operator $\hat{N} = \sum_k \hat{N}_k$ has only one eigenvalue: $N = \frac{1}{\hbar}$. So

$$
\hat{N} = \frac{1}{\hbar}\hat{1} \tag{4.28}
$$

which means that *each state is an eigenstate of $\hat{N}$ and the total number of particles is always $\frac{1}{\hbar}$*. Now it is clear why we are not able to define annihilation or creation operators in $\mathcal{F}_\hbar$. This is because the annihilation of any particle implies the creation of another one in such a way that the number of particles is conserved and is equal to $\frac{1}{\hbar}$. Of course, the vectors $e_{(s_1,s_2,\ldots)}$ are the eigenvectors of the Hamiltonian (4.24). Namely

$$\hat{H}e_{(s_1,s_2,\ldots)} = \hbar\left(\sum_{k\neq 0} s_k\omega_k + \left(\frac{1}{\hbar} - \sum_{k\neq 0} s_k\right)\omega_0\right)e_{(s_1,s_2,\ldots)}. \tag{4.29}$$

It follows that *the ground state of the field is the state with all $\frac{1}{\hbar}$ particles occupying the lowest one-particle state ψ_0 of the energy ε_0*. So the energy of the ground state is

$$E_0 = \frac{1}{\hbar}\varepsilon_0 \tag{4.30}$$

and this corresponds to the Bose–Einstein condensation. As we have decided to identify the functions on $\mathbb{C}P^\infty$ with the Wick symbols rather than with the Berezin ones (see (4.22)) we must use the $*'_B$-product and not the $*_B$-product. Consequently, if $F(z,\overline{z})$ and $G(z,\overline{z})$ are restrictions to U_0 of two functions on $\mathbb{C}P^\infty$ which correspond to the field operators $\hat{F}$ and $\hat{G}$, respectively, then the function (the Wick symbol) corresponding to the product $\hat{F}\hat{G}$ is given on U_0 by (compare with (2.48))

$$\begin{aligned} F(z,\overline{z}) *'_B G(z,\overline{z}) &= c(\hbar)\int_{U_0} F(v,\overline{z})G(z,\overline{v})\exp\left\{\frac{1}{\hbar}\mathcal{K}(z,\overline{z};v,\overline{v})\right\}\mathrm{d}\mu\,(v,\overline{v}) \\ &= G(z,\overline{z}) *_B F(z,\overline{z}). \end{aligned} \tag{4.31}$$

In order to consider the star product (4.31) as a formal one (as it is done in the usual formal deformation quantization) we must expand the right-hand side of (4.31) in the formal series in powers of $\hbar$. This procedure has been developed in the paper by Reshetikhin and Takhtajan [21]. One can easily observe that their normalized $*$-product (see equation (4.6) of [21]) is, in the present case, exactly the Berezin–Wick $*_B$-product given by equation (4.19) because the unit element $e_\hbar(z,\overline{z})$ defined by equation (4.2) in [21] is equal to the normalization factor $c(\hbar)$ (see equation (4.8)) given in this paper. Therefore, using the results of [21] and also the formulae

$$c(\hbar) = \hbar^n\frac{\Gamma(\frac{1}{\hbar}+n+1)}{\Gamma(\frac{1}{\hbar})} = (1+n\hbar)(1+(n-1)\hbar)\ldots(1+\hbar)1 + \hbar\frac{n(n+1)}{2} + \mathrm{O}(\hbar^2)$$

and

$$A := \frac{1}{2}\sum_{j,i\neq 0} g^{\overline{j}i}\frac{\partial^2}{\partial\overline{z}^j\partial z^i}\ln[\det(g_{k\overline{l}})] = -\frac{n(n+1)}{2}$$

one quickly finds that

$$F(z,\overline{z}) *'_B G(z,\overline{z}) = G(z,\overline{z}) *_B F(z,\overline{z}) = GF + \hbar\sum_{j,i\neq 0} g^{\overline{j}i}\frac{\partial G}{\partial\overline{z}^j}\frac{\partial F}{\partial z^i} + \mathrm{O}(h^2). \tag{4.32}$$

We must note that in the case when $n\to\infty$, the formal expansion (4.32) contains divergent terms. Consequently, to avoid this problem we use the strict integral formula for the $*'_B$-product rather than the formal one.

From (4.32) we immediately find that the *Berezin–Wick bracket* defined by

$$\{F,G\}'_B := \frac{1}{\mathrm{i}\hbar}(F *'_B G - G *'_B F) \tag{4.33}$$

reads

$$\{F,G\}'_B = \{F,G\} + \mathrm{O}(\hbar) \tag{4.34}$$

where $\{F,G\}$ is the Poisson bracket of F and G

$$\{F,G\} = \sum_{k,l\neq 0}\omega^{\overline{k}l}\left(\frac{\partial F}{\partial\overline{z}^k}\frac{\partial G}{\partial z^l} - \frac{\partial F}{\partial z^l}\frac{\partial G}{\partial\overline{z}^k}\right). \tag{4.35}$$

5. Wigner functions

In this section we are going to find Wigner functions $\rho_{(s_1,s_2,\ldots)}(z,\overline{z})$ corresponding to the states $e_{(s_1,s_2,\ldots)}$. In terms of the Berezin–Wick $*'_B$-product equation (4.27) reads

$$
\begin{aligned}
&(N_k *'_B \rho_{(s_1,s_2,\ldots)})(z,\overline{z}) = s_k\rho_{(s_1,s_2,\ldots)}(z,\overline{z}), \qquad k \neq 0,\\
&(N_0 *'_B \rho_{(s_1,s_2,\ldots)})(z,\overline{z}) = \left(\frac{1}{\hbar} - \sum_{k\neq 0} s_k\right)\rho_{(s_1,s_2,\ldots)}(z,\overline{z}).
\end{aligned} \tag{5.1}
$$

Employing (4.26) and (4.31) after some work one finds that the system of equations (5.1) is equivalent to the following system of differential equations:

$$
\overline{z}^k \frac{\partial}{\partial \overline{z}^k}[(1+z\overline{z})^{\frac{1}{\hbar}}\rho_{(s_1,s_2,\ldots)}(z,\overline{z})] = s_k[(1+z\overline{z})^{\frac{1}{\hbar}}\rho_{(s_1,s_2,\ldots)}(z,\overline{z})], \qquad k\neq 0. \tag{5.2}
$$

(There is no summation over k!)

The unique real solution normalized by $\mathrm{Tr}(\rho_{(s_1,s_2,\ldots)})(z,\overline{z}) = 1$, where Tr is defined by equation (4.17), reads

$$
\rho_{(s_1,s_2,\ldots)}(z,\overline{z}) = \frac{\frac{1}{\hbar}!}{(\frac{1}{\hbar} - \sum_{k\neq 0} s_k)!(1+z\overline{z})^{\frac{1}{\hbar}}}\prod_{k\neq 0}\frac{|z^k|^{2s_k}}{s_k!} = \frac{e_{(s_1,s_2,\ldots)}(z,\overline{z})\overline{e_{(s_1,s_2,\ldots)}(z,\overline{z})}}{(1+z\overline{z})^{\frac{1}{\hbar}}}. \tag{5.3}
$$

Hence, the Wigner function ρ_0 of the ground state is of the form

$$
\rho_0(z,\overline{z}) = \frac{1}{(1+z\overline{z})^{\frac{1}{\hbar}}}. \tag{5.4}
$$

Then it is easy to find that the expected value $\mathrm{Tr}(\hat{F}\hat{\rho}_{(s_1,s_2,\ldots)})$ of any operator $\hat{F}$ in the Hilbert space $\mathcal{F}_\hbar$ in terms of the corresponding Wigner function is given by

$$
\langle\hat{F}\rangle = c(\hbar)^2\int_{U_0}\rho_{(s_1,s_2,\ldots)}(z,\overline{v})F(v,\overline{z})\exp\left\{\frac{1}{\hbar}\mathcal{K}(z,\overline{z};v,\overline{v})\right\}\mathrm{d}\mu\,(z,\overline{z})\,\mathrm{d}\mu\,(v,\overline{v}). \tag{5.5}
$$

Finally, the von Neumann–Liouville evolution equation for a Wigner function $\rho(t;z,\overline{z})$ is given by

$$
\frac{\partial\rho}{\partial t} = \{\langle\hat{\hat{H}}\rangle,\rho\}'_B. \tag{5.6}
$$

6. Final remarks

In this paper we have investigated the second quantization of the Schrödinger field within the deformation quantization formalism. Comparing the considerations of section 2 with the ones of sections 4 and 5 we conclude that the Berezin deformation quantization of the geometric quantum mechanics leads to some results which do not appear at all in the case of the Berezin deformation quantization of the Schrödinger field (i.e. the usual second quantization). For instance, in the former case one gets that:

(i) $\frac{1}{\hbar}$ is a positive integer.

(ii) The number of particles is constant and is equal to $\frac{1}{\hbar}$. Hence, the ground state corresponds to the Bose–Einstein condensation.

(iii) There do not exist the annihilation and creation operators in the Hilbert space $\mathcal{F}_\hbar$ of the quantized system.

This means that the second quantization and the quantization of geometric quantum mechanics are not equivalent to one another.

An interesting question is also what happens if we quantize geometric quantum mechanics corresponding to the nonlinear quantum mechanics a la Weinberg [45]. Although difficulties with nonlinear quantum mechanics seem to be unavoidable (see e.g. [46]), from the geometric point of view such a quantum mechanics is quite natural [31–34]. We are going to study this problem in a separate paper.

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